

Advanced Quantum Mechanics

Monday, July 15, 2026, 15:00 - 17:00, Exam Hall 3 H13 - J12, Blauwborgje 4

The total number of points (excluding the bonus question) is 50. No books/notes are allowed. Useful formulas can be found on the last page. Do not forget to write your name and student number on the first sheet and number pages. Write neatly, if you want to get a grade. Good luck.

1. Transformations of states and operators

- (a) In this problem, we consider transformations described by the unitary operator $\hat{U} = e^{-\frac{i}{\hbar}(p_0\hat{x}+x_0\hat{p})}$. Use the Baker-Campbell-Hausdorff (BCH) formula to factorize $\hat{U}$:

$$\hat{U} = e^{-\frac{i}{\hbar}p_0\hat{x}}e^{-\frac{i}{\hbar}x_0\hat{p}}e^{i\varphi} = e^{-\frac{i}{\hbar}x_0\hat{p}}e^{-\frac{i}{\hbar}p_0\hat{x}}e^{-i\varphi}. \quad (1)$$

Find φ . [4 points]

- (b) The eigenstate of the momentum operator $|p\rangle$: $\hat{p}|p\rangle = p|p\rangle$ transforms into $|p\rangle' = \hat{U}|p\rangle$. Find $|p\rangle'$. [5 points]
- (c) The coordinate operator $\hat{x}$ transforms into $\hat{x}'$. Find $\hat{x}'$. [5 points]

Hint: Use Eq.(1) or the expansion of $e^{\lambda\hat{A}}\hat{B}e^{-\lambda\hat{A}}$ in powers of λ .

2. Two-dimensional harmonic oscillator in magnetic field

The electron moves in the potential $U(x,y) = \frac{m\omega^2}{2}(x^2 + y^2)$. In addition, a uniform magnetic field B is applied along the z axis. Ignore the coupling of the electron spin to the magnetic field, which in the absence of spin-orbit coupling can be discussed separately. The Hamiltonian has the form:

$$\hat{H} = \frac{(\hat{p}_x + \frac{e}{c}A_x)^2}{2m} + \frac{(\hat{p}_y + \frac{e}{c}A_y)^2}{2m} + \frac{m\omega^2}{2}(x^2 + y^2). \quad (2)$$

- (a) Check that the vector potential $\mathbf{A} = (-\frac{By}{2}, \frac{Bx}{2}, 0)$ describes the uniform magnetic field in the z direction. [2 points]
- (b) Prove that the Hamiltonian (2) can be written in the form,

$$\hat{H} = \frac{(\hat{p}_x^2 + \hat{p}_y^2)}{2m} + \mu_B B \hat{l}_z + \frac{m\Omega^2}{2}(x^2 + y^2), \quad (3)$$

where $\mu_B = \frac{e\hbar}{2mc}$ is the Bohr magneton ($\mu_B B = \frac{\hbar\omega_c}{2}$, $\omega_c = \frac{eB}{mc}$ being the cyclotron frequency) and $\hat{l}_z = \frac{1}{\hbar}\hat{L}_z$ is the z-component of the dimensionless orbital momentum. Find the frequency Ω . [4 points]

- (c) Give a general argument why $\hat{l}_z$ commutes with the Hamiltonian (3) or prove $[\hat{l}_z, \hat{H}] = 0$ by direct calculation. [5 points]

- (d) Creation and annihilation operators for the x- and y-oscillators are:

$$\begin{cases} \hat{x} = \sqrt{\frac{\hbar}{2m\Omega}} (\hat{a}_x + \hat{a}_x^\dagger), & \hat{y} = \sqrt{\frac{\hbar}{2m\Omega}} (\hat{a}_y + \hat{a}_y^\dagger), \\ \hat{p}_x = -i\sqrt{\frac{\hbar m\Omega}{2}} (\hat{a}_x - \hat{a}_x^\dagger), & \hat{p}_y = -i\sqrt{\frac{\hbar m\Omega}{2}} (\hat{a}_y - \hat{a}_y^\dagger). \end{cases} \quad (4)$$

Prove that the Hamiltonian (3) can be written in the form:

$$\hat{H} = \hbar\Omega \left(\hat{a}_x^\dagger \hat{a}_x + \hat{a}_y^\dagger \hat{a}_y + 1 \right) - i\mu_B B \left(\hat{a}_x^\dagger \hat{a}_y - \hat{a}_y^\dagger \hat{a}_x \right) \quad (5)$$

(it is sufficient to express $\hat{l}_z$ in terms of the creation and annihilation operators).

[4 points]

Hint: What can be said about the following commutators:

$[\hat{a}_x, \hat{a}_y]$, $[\hat{a}_x, \hat{a}_y^\dagger]$, $[\hat{a}_x^\dagger, \hat{a}_y]$ and $[\hat{a}_x^\dagger, \hat{a}_y^\dagger]$?

- (e) Use the states $|n_x, n_y\rangle$ with n_x and n_y excitation quanta of the x and y oscillators, respectively,

$$\hat{a}_x^\dagger \hat{a}_x |n_x, n_y\rangle = n_x |n_x, n_y\rangle, \quad \hat{a}_y^\dagger \hat{a}_y |n_x, n_y\rangle = n_y |n_x, n_y\rangle, \quad (6)$$

to find the spectrum of the 2D harmonic oscillator,

$$\hat{H}_{2D} = \hbar\Omega \left(\hat{a}_x^\dagger \hat{a}_x + \hat{a}_y^\dagger \hat{a}_y + 1 \right), \quad (7)$$

(the first part of the Hamiltonian (5)). Express the energy in terms of n_x and n_y , and find degeneracy of each energy level. [5 points]

- (f) The second term in Eq.(5), describing the coupling of the orbital momentum to the magnetic field, lifts the degeneracy. Find $\hat{l}_z|1, 0\rangle$ and $\hat{l}_z|0, 1\rangle$. A superposition of the states $|1, 0\rangle$ and $|0, 1\rangle$,

$$|\Psi\rangle = a|1, 0\rangle + b|0, 1\rangle, \quad (8)$$

can be written as the column

$$|\Psi\rangle = \begin{pmatrix} a \\ b \end{pmatrix}. \quad (9)$$

Show that in this two-dimensional subspace $\hat{l}_z$ acts as the Pauli matrix σ_y . Find eigenvalues and eigenvectors of $\hat{l}_z$. Find superpositions of the states $|1, 0\rangle$ and $|0, 1\rangle$, which are eigenstates of the Hamiltonian (5), and the corresponding energies. [5 points]

Hint: What are eigenvalues of σ_y^2 ?

(g) **Bonus question:** Show that the operators

$$\begin{aligned}\hat{j}_1 &= \frac{1}{2} (\hat{a}_x^\dagger \hat{a}_y + \hat{a}_y^\dagger \hat{a}_x), \\ \hat{j}_2 &= \frac{1}{2i} (\hat{a}_x^\dagger \hat{a}_y - \hat{a}_y^\dagger \hat{a}_x), \\ \hat{j}_3 &= \frac{1}{2} (\hat{a}_x^\dagger \hat{a}_x - \hat{a}_y^\dagger \hat{a}_y),\end{aligned}\tag{10}$$

are generators of $SU(2)$ group, the integrals of motion being the Casimir operator $\hat{j}^2 = \hat{j}_1^2 + \hat{j}_2^2 + \hat{j}_3^2$ and $\hat{j}_2$. [5 points]

Hint: Answering this question may take time. Solve other problems first.

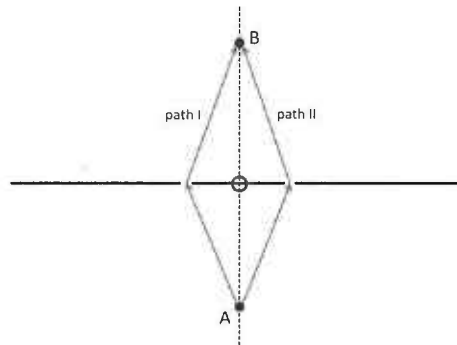


FIG. 1: The double slit setup, which is symmetric under yz mirror (dashed line). Electrons propagate from A to B through two slits in the presence of a solenoid (open circle).

3. Propagator, path integral and Aharonov-Bohm effect

(a) Propagator is the kernel of the evolution operator. For a time-independent Hamiltonian,

$$K(x_2, t_2; x_1, t_1) = \langle x_2 | e^{-\frac{i}{\hbar} \hat{H}(t_2 - t_1)} | x_1 \rangle.\tag{11}$$

Prove

$$\int_{-\infty}^{+\infty} dx_2 K(x_3, t_3; x_2, t_2) K(x_2, t_2; x_1, t_1) = K(x_3, t_3; x_1, t_1).\tag{12}$$

[3 points]

Hint: Completeness relation.

- (b) Consider a double slit experiment, in which electrons propagate from A to B (see Fig. 1). The electron propagator is the sum of two path integrals:

$$K(x_2, t_2; x_1, t_1) = \int D\mathbf{x}_I(t) e^{\frac{i}{\hbar} S[\mathbf{x}_I(t)]} + \int D\mathbf{x}_{II}(t) e^{\frac{i}{\hbar} S[\mathbf{x}_{II}(t)]}, \quad (13)$$

where path I (path II) goes from A to B through the left(right) slit (contributions of other paths to the propagator are neglected). The points A and B lie on the mirror symmetry plane, so that for each path I, there is a path II (mirror image) with the same action: $S[\mathbf{x}_{II}(t)] = S[\mathbf{x}_I(t)]$.

An infinitely thin solenoid carrying the magnetic field flux Φ is placed between the two slits (see Fig. 1). Show that in the presence of solenoid,

$$K(x_2, t_2; x_1, t_1) = e^{i\varphi_I} \int \mathcal{D}\mathbf{x}_I(t) e^{\frac{i}{\hbar} S[\mathbf{x}_I(t)]} + e^{i\varphi_{II}} \int \mathcal{D}\mathbf{x}_{II}(t) e^{\frac{i}{\hbar} S[\mathbf{x}_{II}(t)]}, \quad (14)$$

where the phase φ_I is the same for all paths $\mathbf{x}_I(t)$ and the phase φ_{II} is independent of path $\mathbf{x}_{II}(t)$. Find $\varphi = \varphi_{II} - \varphi_I$. [5 points]

How is the action of a charged particle modified by the magnetic field in solenoid?

Use Stokes' lemma for the area bounded by two paths.

- (c) In the absence of the solenoid, the electron beam intensity measured at point B is $I(0)$. Find the intensity $I(\Phi)$ as a function of the flux Φ in solenoid, in particular, $I(\frac{\Phi_0}{2})$ and $I(\Phi_0)$, where $\Phi_0 = \frac{hc}{e} = \frac{2\pi\hbar c}{e}$ is the flux quantum. [3 points]

Hint: What does mirror symmetry tell us about the relative values of the two path integrals in the absence of solenoid?

Useful formulae

- Commutators:**

$$[\hat{A}, \hat{B}] = -[\hat{B}, \hat{A}], \quad [\hat{A}\hat{B}, \hat{C}] = \hat{A}[\hat{B}, \hat{C}] + [\hat{A}, \hat{C}]\hat{B}, \quad [\hat{A}, \hat{B}\hat{C}] = \hat{B}[\hat{A}, \hat{C}] + [\hat{A}, \hat{B}]\hat{C}$$

- Transformation of states and operators**

$$|\Psi'\rangle = \hat{U}|\Psi\rangle, \quad \hat{A}' = \hat{U}^\dagger \hat{A} \hat{U}, \quad \hat{U}^\dagger \hat{U} = 1$$

$$e^{\lambda \hat{A}} \hat{B} e^{-\lambda \hat{A}} = \hat{B} + \frac{\lambda^1}{1!} [\hat{A}, \hat{B}] + \frac{\lambda^2}{2!} [\hat{A}, [\hat{A}, \hat{B}]] + \dots$$

$$\text{BCH formula: } e^{\hat{A}+\hat{B}} = e^{\hat{A}} e^{\hat{B}} e^{-\frac{1}{2}\hat{C}}, \text{ if } \hat{C} = [\hat{A}, \hat{B}] \text{ commutes with } \hat{A} \text{ and } \hat{B}$$

- Heisenberg equation of motion:** $\frac{d\hat{A}(t)}{dt} = \frac{\partial \hat{A}(t)}{\partial t} + \frac{1}{i\hbar} [\hat{A}(t), \hat{H}]$

- Completeness relations:** $\sum_n |n\rangle \langle n| = \int dx |x\rangle \langle x| = \int \frac{dp}{2\pi\hbar} |p\rangle \langle p| = 1$

- Harmonic oscillator**

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{m\omega^2 \hat{x}^2}{2} = \hbar\omega \left(\hat{a}^\dagger \hat{a} + \frac{1}{2} \right), \quad [\hat{a}, \hat{a}^\dagger] = 1$$

$$\begin{cases} \hat{x} = \sqrt{\frac{\hbar}{2m\omega}} (\hat{a} + \hat{a}^\dagger) \\ \hat{p} = -i\sqrt{\frac{\hbar m\omega}{2}} (\hat{a} - \hat{a}^\dagger) \end{cases}$$

$$\hat{a}|n\rangle = \sqrt{n}|n-1\rangle, \quad \hat{a}^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle, \quad \hat{a}^\dagger \hat{a}|n\rangle = n|n\rangle, \quad |n\rangle = \frac{(\hat{a}^\dagger)^n}{\sqrt{n!}}|0\rangle$$

- Coherent state**

$$|\alpha\rangle = e^{-\frac{|\alpha|^2}{2}} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle, \quad \hat{a}|\alpha\rangle = \alpha|\alpha\rangle$$

- Levi-Civita tensor**

$$\epsilon_{nmk}\epsilon_{kpq} = \delta_{np}\delta_{mq} - \delta_{nq}\delta_{mp}, \quad \epsilon_{nmk}\epsilon_{nml} = 2\delta_{kl}$$

$$\epsilon_{ijk}A_jB_k = [\mathbf{A} \times \mathbf{B}]_i, \quad \epsilon_{ijk}A_iB_jC_k = (\mathbf{A} \cdot [\mathbf{B} \times \mathbf{C}])$$

- Particle in electromagnetic fields**

$$\mathbf{E} = -\frac{1}{c} \frac{\partial \mathbf{A}}{\partial t} - \nabla \phi, \quad \mathbf{B} = [\nabla \times \mathbf{A}], \quad \hat{H} = \frac{(\hat{\mathbf{p}} - \frac{q}{c} \hat{\mathbf{A}})^2}{2m} + q\phi, \quad L = \frac{m\mathbf{v}^2}{2} + \frac{q}{c} \mathbf{v} \cdot \mathbf{A} - q\phi$$

- Angular momentum**

$$[\hat{j}_n, \hat{j}_m] = i\epsilon_{nmk}\hat{j}_k$$

$$\hat{j}^2|jm\rangle = j(j+1)|jm\rangle, \quad \hat{j}_z|jm\rangle = m|jm\rangle,$$

$$\hat{j}_+|jm\rangle = \sqrt{(j-m)(j+m+1)}|j, m+1\rangle, \quad \hat{j}_-|jm\rangle = \sqrt{(j+m)(j-m+1)}|j, m-1\rangle$$